Suggested solution of test 1

1. Let $P(n)$ be the following statement: For any $a \le b$, $a, b \in \mathbb{R}$, if $[a,b] \subset \cup_{k=1}^{n} I_k$ where I_k are open intervals, then
$$\sum_{k=1}^{n} |I_k| \ge b - a.$$
When $n = 1$, the conclusion is immediate. Assume $P(N)$ is true for some $N \in \mathbb{N}$, let $a \le b$, $a, b \in \mathbb{R}$ and
$$[a,b] \subset \bigcup_{k=1}^{N+1} I_k.$$
Say for example, $a \in I_1 = (x_1, x_2)$. Then
$$[x_2, b] \subset \bigcup_{k=2}^{N+1} I_k.$$
By induction hypothesis, $b - x_2 \le \sum_{k=2}^{N+1} |I_k|$. Hence
$$b - a = b - x_2 + x_2 - a \le b - x_2 + x_2 - x_1 \le \sum_{k=1}^{N+1} |I_k|.$$
By MI, it is done.

2. Recall
$$m^*([a,b]) = \inf\{\sum_{k=1}^{\infty} |I_k| : [a,b] \subset \cup I_k\}.$$
Since $[a,b] \subset (a-\epsilon, b+\epsilon)$ for all $\epsilon > 0$, we have
$$m^*([a,b]) \le b - a.$$
Let I_k be a open cover of $[a,b]$, by compactness, there exists a finite subcover $\{I_j\}_{j=1}^{N}$ of $[a,b]$. By Q1,
$$b - a \le \sum_{j=1}^{N} |I_j| \le \sum_{k=1}^{\infty} |I_k|.$$
Hence, $m^*([a,b]) \ge b - a$.

3. f is called continuous if $\forall x \in \mathbb{R}$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that for all $|x - y| < \delta$,
$$|f(x) - f(y)| < \epsilon.$$
If G is non-empty open set on $\mathbb{R}$, let $x \in f^{-1}(G)$, $f(x) = y \in G$. Then $\exists \epsilon > 0$ such that
$$V_\epsilon(f(x)) \subset G.$$
By continuity, $\exists \delta > 0$ such that whenever $z \in V_\delta(x)$,
$$|f(z) - f(x)| < \epsilon.$$
That is to say $V_\delta(x) \subset f^{-1}(G)$.

4. Denote $\mathcal{B}$ to be the intersection of all σ-algebra containing all open sets. It is nonempty as $P(\mathbb{R})$ is a σ algebra containing the open sets. It is clear that $\mathcal{B}$ contains open sets of $\mathbb{R}$. It suffices to show that it is a σ-algebra. The minimality follows easily from the construction.

   For simplicity, we denote Σ to be a generic σ-algebra containing open sets. Suppose $A \in \mathcal{B} \subset \Sigma$ for any Σ. Then $A^c \in \Sigma$ for all Σ. Hence $A^c \in \mathcal{B}$.

   If A_n, $n \in \mathbb{N}$ is inside $\mathcal{B}$, $A_n \in \Sigma$ for all Σ which implies $\cup A_n \in \Sigma$ for any Σ. Therefore, $\cup A_n \in \mathcal{B}$. Of course $\emptyset \in \mathcal{B}$ as $\emptyset \in \Sigma$ for any Σ.

5. Let $M = \{A \in P(\mathbb{R}) : f^{-1}(A) \in \mathcal{B}\}$. By Q3, it contains all open sets. And using the fact that
$$f^{-1}(A^c) = (f^{-1}(A))^c$$
   and
$$\bigcup_{k=1}^{\infty} f^{-1}(A_k) = f^{-1}\left(\bigcup_{k=1}^{\infty} A_k\right).$$
   M is a σ-algebra containing open sets. In particular, $\mathcal{B} \subset M$ by the minimality.